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An Extension of E. Bishop's Localization Theorem

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We prove that if f is a function belonging to Baire first class on a compact set $K \subset \mathbb{C}$ and each point of K has a (closed) neighborhood where f is the pointwise limit of some sequence of uniformly bounded rational functions, then f on the whole of K is the pointwise limit of a sequence of rational functions uniformly bounded on K. This is an extension of Bishop's localization theorem. As an application we establish a "pointwise" version of Mergelyan's classical theorem on uniform approximation by rational functions on compact sets for which the components of its complement have diameters greater than a fixed positive number. © 2001 Academic Press

1. INTRODUCTION AND THE MAIN RESULT

The purpose of this article is to establish an extension of E. Bishop's localization theorem that is valid for the case of pointwise bounded approximation, and to give as an application of this extension the solution to a related approximation problem.

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For a given set S we denote by S^0, $\overline{S}$, and ∂S the set of interior points, the closure, and the boundary of S, respectively. Let K be an arbitrary compact set in the complex plane. As usual we denote by $C(K)$ the space of all functions continuous on K; by $A(K)$ all functions of $C(K)$ that are analytic in K^0; by $R(K)$ all functions of $A(K)$ that are uniform limits on K of rational functions with poles outside K; and by $\|f\|$ the sup norm of f on K.

The following classical localization theorem in complex approximation is due to Bishop.

THEOREM A (E. Bishop, 1958). *Let $f \in C(K)$ be such that every point $z \in K$ has a neighborhood U_z with the property that the restriction $f|_{K \cap \bar{U}_z}$ of f to $K \cap \bar{U}_z$ is in $R(K \cap \bar{U}_z)$. Then f is in $R(K)$.*

There is an extensive literature on this remarkable result (see for example, [8, 9, 11, 14, 21]).

Denote by $B(K)$ all bounded functions on K belonging to *Baire first class*, that is

$$B(K) = \{f\colon f(z) \text{ is bounded on } K \text{ and there exist } f_n(z) \in C(K) \text{ such that } \lim_{n \to \infty} f_n(z) = f(z) \text{ on } K\}.$$

We also set

$$BR(K) = \{f \in B(K) : \text{there exist rational functions } r_n \in R(K) \text{ which are uniformly bounded on } K \text{ and } \lim_{n \to \infty} r_n(z) = f(z) \text{ on } K\}. \tag{1}$$

Evidently $R(K) \subset BR(K) \subset B(K)$. We now formulate the main result of this article.

THEOREM 1. *Let $f \in B(K)$ be such that every point $z \in K$ has a neighborhood U_z with the property that the restriction $f|_{K \cap \bar{U}_z}$ is in $BR(K \cap \bar{U}_z)$. Then $f \in BR(K)$; more precisely, there exist rational functions $r_n \in R(K)$ such that $\|r_n\| \leqslant \|f\|$ for all n and $\lim_{n \to \infty} r_n(z) = f(z)$ on K. In addition, if $f \in C(K)$, then $f \in R(K)$.*

In other words, if every point $z \in K$ has a neighborhood $K \cap \bar{U}_z$ where f is the pointwise limit of a uniformly bounded (on $K \cap \bar{U}_z$) sequence of rational functions, then on the whole of K the function f is the pointwise limit of a sequence of rational functions uniformly bounded (by $\|f\|$) on K.

Remark 1. Obviously pointwise bounded convergence on a compact set is implied by uniform convergence on that compact set. Therefore, under the hypothesis of Bishop's theorem, the hypothesis of Theorem 1 holds and the last sentence of Theorem 1 shows that it actually contains Bishop's theorem. We note, however, that both the proofs of Bishop's theorem and Theorem 1 rely on Bishop's Splitting lemma.

Problems of pointwise approximation by polynomials and rational (or analytic) functions in the complex plane were investigated by many authors including W. F. Osgood, P. Montel, C. Carathéodory, F. Hartogs, M. A. Lavrentyev, M. V. Keldysh, and S. N. Mergelyan (see [3, 15, 16], and references therein). Further results on this topic can be found in [4–7, 9, 10, 12, 18, 19].

In a well known series of investigations P. Ahern and D. Sarason [1], T. W. Gamelin, J. B. Garnett, and A. M. Davie [6, 7, 10] considered problems of pointwise bounded approximation by rational functions (by elements of $R(K)$) of functions defined on the open set of interior points K^0 of compact K; some of these results are also presented in Gamelin's book [9]. The novelty of the present article is that we consider the problem of pointwise bounded approximation by elements of $R(K)$ on an arbitrary compact set K including its boundary.

If $f \in BR(K)$, then, trivially, the hypotheses of Theorem 1 are satisfied and so the more precise conclusion of the theorem asserts that there exist rational functions $r_n \in R(K)$ such that $\|r_n\| \leqslant \|f\|$ for all n and $\lim_{n\to\infty} r_n(z) = f(z)$ on K. That is, if pointwise bounded rational approximation is possible, then as a bound for approximating rational functions one can always achieve the smallest possible bound $\|f\|$. Note that for the case of approximation on K^0 the solution of the corresponding problem requires much more resources (see [9], Chap. 8, Theorem 11.1, and [6]).

Remark 2. Concerning Theorem 1, it has been pointed out by a referee that it is possible to provide an alternative proof for the fact that $f \in BR(K)$ by using a suitable variation of a constructive method of Garnett involving a partitioning of unity (cf. [21], p. 97). However, this method doesn't lead to the precise estimate (namely $\|f\|$) for the norms of the approximating rational functions. As shown in Section 4 an advantage of our method is that it yields this estimate via a straightforward argument.

One can formulate a general problem as follows: under what conditions on K and $f \in B(K)$ it is true that $f \in BR(K)$? As an application of Theorem 1 (see Theorem 2 below) we give a complete solution of this problem by providing an explicit description of the class $BR(K)$ in the case when each component of the complement of K has a diameter greater than a fixed positive number δ. In fact Theorem 2 is a "pointwise bounded version" of Mergelyan's classical theorem [17] (see also [9], Chap. 2, Theorem 10.4) on uniform approximation by rational functions, which states that $A(K) = R(K)$ for an arbitrary compact K having the mentioned property.

Theorem 2 can also be compared with a theorem of Ahern and Sarason [1] (see also [9] Chap. 6, Theorem 5.3) and its generalization (see [9], Chap. 8, Corollary 10.6), asserting that each bounded analytic function on K^0 is a pointwise limit of a uniformly bounded (on K) sequence from $R(K)$. Theorem 2 provides an answer to a natural question arising in connection with this result.

Bishop's Splitting lemma (see for example [14]) and a few other facts from functional analysis will be used in our proofs. Generally it is well known that functional analysis has important applications for pointwise approximation problems. In the proof of Theorem 1 we apply, in particular, some arguments of S. V. Kolesnikov [12] and A. M. Davie [6]. We remark that the basic approach using the concept of weak convergence in normed spaces was in fact already applied to pointwise approximation problems in 1961 by S. N. Mergelyan [16].

The formulation of Theorem 2 involves an appropriate term from conformal mapping suggested by Lavrentyev [15] for pointwise approximation problems on general sets of the complex plane. The connection of this concept of Lavrentyev with potential theory was explored in an article of M. Brelot [2].

2. AUXILIARY RESULTS AND DEFINITIONS

Let K and $R(K)$ be as above and μ be a complex Borel measure on K. As usual we say that μ is orthogonal to $R(K)$ if $\int f\,d\mu = 0$ for each $f \in R(K)$.

THEOREM B (Bishop's Splitting lemma). *Let μ be a complex Borel measure on K orthogonal to $R(K)$ and let $\{U_n\}$ be a finite covering of K. Then there exist measures μ_n such that $\mu = \sum \mu_n$, $\operatorname{supp}(\mu_n) \subset K \cap U_n$, and μ_n is orthogonal to $R(K \cap \bar{U}_n)$.*

Let E be an arbitrary compact set whose complement is connected. Let $\{D_n\}$ be the connected components of E^0; $E^0 = \bigcup_{n=1}^{\infty} D_n$. Each D_n is simply connected and there exists a conformal mapping $z = \psi_n(w)$ of the open unit disk U in the w-plane onto D_n. Let $S_n \subset \partial U$ be the set of all points where $\psi_n(w)$ has nontangential limits. Evidently S_n is a Borel set and by Fatou's theorem its Lebesgue measure is equal to 2π. We define the function $\psi_n(w)$ by its boundary values also on the set S_n. Then $\psi_n(w)$ is a Borel function on S_n and the points of the set $\psi_n(S_n)$ are accessible boundary points of the domain D_n. If $f(z)$ is a function defined on ∂D_n, then one can consider the function $f(\psi_n(w))$ defined a.e. on ∂U. The following definition is due to Lavrentyev [15].

DEFINITION 1. Let $f(z)$ be a function defined on ∂D_n and let $\Phi(z)$ be a bounded analytic function on D_n. We say that the boundary values of $\Phi(z)$ are equal to $f(z)$ almost everywhere on ∂D_n in the conformal mapping sense if the angular boundary values of the function $\Phi(\psi_n(w))$ coincide almost everywhere on the unit circumference ∂U with the values of the function $f(\psi_n(w))$.

We will use the following result from [4], Theorem 2. Its necessity part was proved earlier by Lavrentyev [15], Theorem 15.

THEOREM C. *Let E be an arbitrary compact set whose complement is connected and let f be bounded and belong to Baire first class on E (that is, $f \in B(E)$). In order that there exist a polynomial sequence uniformly bounded on E converging to $f(z)$ for each $z \in E$, it is necessary and sufficient that the restriction $\Phi(z) := f(z)|_{E^0}$ ($E^0 = \bigcup_{n=1}^{\infty} D_n$) be analytic and for each $n = 1, 2, \ldots$ the boundary values of $\Phi(z)$ are equal to $f(z)$ almost everywhere on ∂D_n in the conformal mapping sense.*

3. AN APPLICATION OF THE MAIN RESULT

Let K be an arbitrary compact set such that each component of the complement of K has diameter greater than a given positive number δ. One can cover K by a finite collection of open disks $\{U_j\}_{j=1}^{l}$ such that each set $E_j := K \cap \bar{U}_j$, $j = 1, 2, \ldots, l$, has a connected complement. Indeed, each finite cover of K consisting of disks with diameters less than δ has the mentioned property.

As an application of Theorem 1 we will prove the following result.

THEOREM 2. *Let K be an arbitrary compact set such that each component of the complement of K has diameter greater than a given positive number δ and let $f \in B(K)$. In order that $f \in BR(K)$, it is necessary and sufficient that the function $\Phi(z) := f(z)|_{K^0}$ be analytic on K^0 and there exist open disks $\{U_j\}_{j=1}^{l}$ covering K such that the set $E_j := K \cap \bar{U}_j$ has a connected complement and for each component $D_n^{(j)}$ of E_j^0 ($E_j^0 = \bigcup_{n=1}^{\infty} D_n^{(j)}$) the boundary values of $\Phi(z)$ are equal to $f(z)$ almost everywhere on $\partial D_n^{(j)}$ in the conformal mapping sense ($j = 1, 2, \ldots, l$).*

Let $\Phi(z)$ be a bounded and analytic function on K^0. Then by a known theorem of Ahern and Sarason and its generalization (see Section 1) the function $\Phi(z)$ on K^0 is the pointwise limit of rational functions $\{r_m(z)\}$ uniformly bounded on K. One can ask the following natural question: *Under the conditions of this theorem, what additional assumption on the function $\Phi(z)$ will further imply that the sequence $\{r_m(z)\}$ also converges on the boundary ∂K?*

As a direct consequence of Theorem 2 one can give a complete answer of this question as follows:

Let K be an arbitrary compact set such that each component of the complement of K has diameter greater than some positive number δ and let $\Phi(z)$ be a bounded and analytic function on K^0. There exists a sequence $\{r_m(z)\}$ of rational functions uniformly bounded on K that converges on K^0 to $\Phi(z)$, and also converges on ∂K, if and only if there exists a bounded function $f(z) \in B(K)$ such that $f(z) = \Phi(z)$ on K^0 and $f(z)$ satisfies the conditions of Theorem 2.

4. PROOFS

Proof of Theorem 1. Because K is a compact set, the hypothesis of the theorem implies that there exists a finite open covering $\{U_j\}$, $j = 1, 2, ..., l$, of K such that each U_j is a disk centered at a point of K and the restriction $f|_{K \cap \overline{U_j}}$ is in $BR(K \cap \overline{U_j})$. By (1) this means that there exist rational functions $r_{m,j} \in R(K \cap \bar{U}_j)$ such that

$$|r_{m,j}(z)| \leqslant M_j \quad \text{and} \quad \lim_{m \to \infty} r_{m,j}(z) = f(z) \qquad \text{on} \quad K \cap \overline{U_j} \tag{2}$$

for some $M_j > 0$, $j = 1, 2, ..., l$.

Let μ be a complex Borel measure on K orthogonal to $R(K)$. By Theorem B we have

$$\mu = \sum \mu_j, \qquad \operatorname{supp}(\mu_j) \subset K \cap U_j \qquad \text{and } \mu_j \text{ is orthogonal to } R(K \cap \overline{U_j}).$$

Therefore,

$$\int f\, d\mu = \sum \int f\, d\mu_j$$

and because μ_j is orthogonal to $R(K \cap \overline{U_j})$ it follows from (2) by Lebesgue's theorem that

$$\int f\, d\mu_j = \lim_{m \to \infty} \int r_{m,j}\, d\mu_j = 0, \qquad j = 1, 2, ..., l.$$

Hence for each μ that is orthogonal to $R(K)$ we have $\int f\, d\mu = 0$. Because f is bounded and belongs to Baire first class ($f \in B(K)$) one can find a sequence $\{f_m(z)\}$ of continuous functions uniformly bounded on K such that $\lim f_m(z) = f(z)$ on K. We can even assume $\|f_m\| \leqslant \|f\|$ since otherwise one can simply replace $f_m(z)$ by $g(f_m(z))$, where $g(z)=z$ if $|z| \leqslant \|f\|$ and $g(z)=(z\,\|f\|)/|z|$ if $|z| > \|f\|$. Again by Lebesgue's theorem,

$$\lim_{m \to \infty} \int f_m \, d\mu = \int f \, d\mu = 0. \tag{3}$$

By the Riesz theorem, a continuous linear functional L on $C(K)$ has the representation

$$L(g) = \int g \, d\lambda,$$

where λ is some complex Borel measure on K. Therefore a continuous linear functional ϕ on the quotient space $C(K)/R(K)$ can be represented as follows (cf. [13], p. 192, and [12] for the particular case when K is the unit disk),

$$\phi(g + R(K)) = \int g \, d\mu,$$

where $g + R(K)$ is an element of the space $C(K)/R(K)$ (that is $g + R(K) = \{g + u : u \in R(K)\}$) and μ is orthogonal to $R(K)$. Consequently, (3) implies that

$$\lim_{m \to \infty} \phi(f_m + R(K)) = 0.$$

This means that the sequence $\{f_m + R(K)\}$ converges weakly to zero in the quotient space $C(K)/R(K)$ and we can now apply arguments similar to those used in [12]. By Mazur's theorem (see [20], p. 120) there exist finite convex linear combinations of elements of $\{f_m + R(K)\}$ that converge to zero in the norm of the quotient space. Hence for each $\varepsilon > 0$ there exists a linear combination

$$\sum_{m=1}^{k} v_m(f_m + R(K)) = \sum_{m=1}^{k} v_m f_m + R(K); \qquad \sum_{m=1}^{k} v_m = 1, \quad v_m \geqslant 0,$$

with a quotient norm less than ε. Therefore, by the definition of the quotient norm there exist a function $u \in R(K)$ such that

$$\max_{z \in K} \left| \sum_{m=1}^{k} v_m f_m(z) - u(z) \right| < \varepsilon.$$

We can apply the same argument to the sequence $\{f_j(z)\}_{j=m}^{\infty}$ and $\varepsilon = 1/m$ which will give a corresponding function $u_m(z) \in R(K)$, $m = 1, 2, \ldots$. It is easy to see that along with $\{f_m(z)\}$, the sequence $\{u_m(z)\}$ also is uniformly bounded ($\|u_m\| \leqslant \|f\| + 1/m$ since $\|f_m\| \leqslant \|f\|$) and converges to $f(z)$ on K. For $u_m(z) \in R(K)$ there exists a rational function $r_m(z) \in R(K)$ such that $|u_m(z) - r_m(z)| < 1/m$ on K. The rational functions $r_m(z)\,\|f\|/(\|f\| + 2/m)$ converge to $f(z)$ on K and are uniformly bounded by $\|f\|$ as claimed in the theorem.

If $f(z)$ is continuous, in the above proof one can choose continuous functions $f_m(z)$ by simply setting $f_m(z) = f(z)$. This means that $\sum_{m=1}^{k} v_m f_m(z) = \sum_{m=1}^{k} v_m f(z) = f(z)$, $z \in K$, and clearly the sequence $\{r_m(z)\}$ in the above argument converges to $f(z)$ uniformly on K.

Proof of Theorem 2 (Necessity). In fact the proof of necessity follows from the previously mentioned theorem of Lavrentyev [15] (Theorem 15). Clearly the function $\Phi(z) = f(z)|_{K^0}$ is analytic on K^0. Fix an open disk covering $\{U_j\}_{j=1}^{l}$ of K such that each $E_j = K \cap \overline{U_j}$ has a connected complement (as we already noted for K from Theorem 2 such a covering always exists). Because E_j, $(j = 1, 2, ..., l)$ has a connected complement we can approximate on it the rational function $r_m(z)$ (which we have by condition of theorem) uniformly by a polynomial $P_{m,j}$ with error at most $1/m$. Now the sequence of polynomials $\{P_{m,j}\}$ is uniformly bounded on E_j and converges to $f|_{E_j}$ on E_j. By the necessity part of Theorem C (which is the same as Lavrentyev's noted theorem) the proof is done.

(Sufficiency). Each E_j has a connected complement and all the conditions of Theorem C are satisfied. Therefore by that theorem there exists a polynomial sequence $\{P_{n,j}(z)\}$ uniformly bounded on E_j converging to $f(z)$ on E_j $(j = 1, 2, ..., l)$. This means that the restriction $f|_{K \cap \bar{U}_j}$ is in $BR(K \cap \overline{U_j})$, $j = 1, 2, ..., l$. Each $z \in K$ belongs to some U_j and one can find an open circle (centered at z) U_z, such that $z \in U_z \subset \overline{U_z} \subset U_j$. Clearly the restriction $R(K \cap \overline{U_j})|_{K \cap \overline{U_z}}$ is a subclass of $R(K \cap \overline{U_z})$ and $f|_{K \cap \overline{U_z}}$ coincides with $f|_{K \cap \overline{U_j}}$ on $K \cap \overline{U_z}$. Therefore $f|_{K \cap \overline{U_z}}$ is in $R(K \cap \overline{U_z})$. The sufficiency now follows from Theorem 1.

Remark 3. In connection with Theorem 1, Professor D. Gaier has posed the following problem: Does there exist a localization theorem for pointwise approximation by a sequence of rational functions which is unbounded on K? This natural question will be addressed in a future publication.

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